



Open universal sets

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Abstract

All spaces are assumed to be regular Hausdorff topological spaces. Let X and Y be spaces. An open subset U of $X \times Y$ is said to be an open universal set for X parametrised by Y if for all open V in X there is an element y of Y such that $V = \{x: (x, y) \in U\}$.

If X has an open universal set parametrised by Y and $n \in \omega$, then $w(X) \leq nw(Y)$, $hd(X^n) \leq hL(Y^n)$, $hL(X^n) \leq hd(Y^n)$ and $hc(X^n) \leq hc(Y^n)$. If X is also compact, then $hL(X^n) \leq hL(Y^n)$ and $hd(X^n) \leq hd(Y^n)$. If X has a G_δ -diagonal, then $hd(X^\omega) \leq hL(Y)$, $hL(X^\omega) \leq hd(Y)$ and $hc(X^\omega) \leq hc(Y)$.

The statement ‘every compact zero-dimensional space with an open universal set parametrised by a space with the hereditary c.c.c. is metrisable’ is consistent and independent of ZFC. The statement ‘every cometrisable space with an open universal parametrised by a hereditarily c.c.c. space is metrisable’ is consistent and independent.

Relevant examples are presented.

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An open set in the product space $X \times Y$ is said to be an open universal set of X parametrised by Y if every open subset V of X is of the form $U^y = \{x \in X: (x, y) \in U\}$ for some $y \in Y$. Universal sets have been considered previously in [8–10], where the existence of sets parametrising countable closed sets, countable G_δ -sets or compact sets were considered. Considerations of universal sets of higher Borel classes turned out to produce a hierarchy of results and examples, and these are recorded by the authors in [4]. We concern ourselves with open universal sets in this paper, giving special attention to the

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cases when X has a G_δ -diagonal or when X is compact. The cardinal invariants, hereditary Lindelöf degree, hL , hereditary density, hd , hereditary cellularity, hc , and weight, w , are looked at as in [4], as well as metrisability.

Section 1 lays out the definitions, notations and basic results for use in the subsequent sections. Section 2 contains results shared by all (regular Hausdorff) spaces, relating their properties with those of the parametrising space. For example, if X has an open universal set parametrised by Y , then $hd(X) \leq hL(Y)$ and $hL(X) \leq hd(Y)$. In Sections 3 and 4 we shall note the stronger theorems obtained as we put extra properties on the space X (diagonal properties and compactness respectively). Continuing the example, if X has a G_δ -diagonal, then we know that $hd(X^n) \leq hL(Y)$ and $hL(X^n) \leq hd(Y)$, for any $n \in \omega$; while if X is compact, then $hd(X^n) \leq hd(Y^n)$ and $hL(X^n) \leq hL(Y^n)$ hold for any $n \in \omega$.

1. Definitions and useful results

All spaces are regular Hausdorff topological spaces unless stated otherwise. Our topological notation follows that of [3].

1.1. Cardinal invariants

Let X be a space. The *weight*, $w(X)$ (respectively *netweight*, $nw(X)$), of X is the minimal size of a base (respectively network) for X . The *hereditary density*, $hd(X)$ (respectively *hereditary Lindelöf degree*, $hL(X)$; respectively *hereditary cellularity*, $hc(X)$), of X , is the suprema of all cardinals κ , such that there are distinct $x_\alpha \in V_\alpha$ open ($\alpha \in \kappa$) with the property that $x_\beta \in V_\alpha$ implies $\beta \geq \alpha$ (respectively implies $\beta \leq \alpha$; respectively implies $\alpha = \beta$). A space X is *hereditarily separable* if $hd(X) \leq \aleph_0$, is *hereditarily Lindelöf* if $hL(X) \leq \aleph_0$, or has the *hereditary c.c.c.* if $hc(X) \leq \aleph_0$.

1.2. S - and L -spaces

A (Hausdorff and regular) space is an S -space (respectively *strong S -space*) if it is hereditarily separable (respectively hereditarily separable in all finite powers) but not hereditarily Lindelöf. A (Hausdorff and regular) space is an L -space (respectively *textitstrong L -space*) if it is hereditarily Lindelöf (respectively hereditarily Lindelöf in all finite powers) but not hereditarily separable. We note that if a space has the hereditary c.c.c. (or hereditarily Lindelöf, or hereditarily separable) in each of its finite powers, then the ω th power of the space has the same property. We further note that the product of a hereditarily Lindelöf (or hereditarily separable) space with a second countable space has the same property.

The statement: (S) ‘There are no S -spaces’ is consistent [12,11]. As is a related statement: (B) ‘A Boolean algebra with no uncountable weak antichains is countable’ [1,2].

There are many (consistent) examples of S - and L -spaces. We will make use of the following constructions of Todorćević [13].

Orders on ω^ω : Define orders on ω^ω as follows: $f <^* g$ if for all but finitely many n , $f(n) < g(n)$; $f \leq g$ if for all n , $f(n) \leq g(n)$; and $f \leq_{lex} g$ if f precedes g in the

lexicographic order. The cardinal \mathfrak{b} is the supremum of all cardinals κ such that there is a subset of ω^ω of order type κ under $<^*$.

Todorćević's examples: Let X be any subset of the real line, and let τ be a topology on X finer than the Euclidean topology. We will call (X, τ) a *Kunen line type space*, if every τ -closure of a subset of X differs from the Euclidean closure of X only by countably many elements. It is easy to see that any Kunen line type space is hereditarily separable. The Kunen line is of Kunen line type (!), as is the Sorgenfrey topology on any subset of the reals.

Assume $\mathfrak{b} = \omega_1$, and fix a subset, A , of ω^ω of order type ω_1 under $<^*$. The set A is a λ -subset of \mathbb{R} (an uncountable subset of the real line such that all countable subsets are relative G_δ -sets). In Chapter 2 of his book [13], Todorćević associates to a certain type of map, H , of A into the countable subsets of A a space $A[H]$ with the following properties: the topology on $A[H]$ is of Kunen line type, and $A[H]$ is a locally compact strong S -space.

In Chapter 0 of the same monograph, for any partial order \leq on A , Todorćević defines the space $A[\leq]$ with basic open neighbourhoods of a in A of the form, for $n \in \omega$, $B_{n,a}[\leq a] = \{b \in A: b \leq a\} \cap \{b \in A: a \upharpoonright n = b \upharpoonright n\}$. Clearly the topology on $A[\leq]$ refines the Euclidean topology. Todorćević proves (Theorem 0.6) that $A[\leq]$ is a strong S -space and $A[\geq]$ is a strong L -space. He also shows (Theorem 3.0) that $A[\leq_{lex}]$ and $A[\geq_{lex}]$ (which are homeomorphic to subsets of the Sorgenfrey line) are hereditarily separable and hereditarily Lindelöf in all finite powers.

We now argue that $A[\leq]$ is a Kunen line type space, because the topology on $A[\leq]$ is contained in that of $A[H]$ (which is of Kunen line type), provided the function H is chosen carefully. On p. 23 of Todorćević's monograph, $H(b)$ is defined as a subset of the set of all a in A such that $a <^* b$. However on p. 24, he observes, 'Note that ... we could have added the condition $a \leq b$ in the definition of H instead of $a <^* b$. But since we don't have a use for this, we keep the definition as it is'. Assuming H does satisfy this stronger property, then in the next few lines, when the $A[H]$ topology is defined, it is clear that the n th basic neighbourhood of a element b of A , is contained in $B_{n,a}[\leq a]$.

1.3. Open universal sets

Definition 1. Let X and Y be spaces. An open (respectively closed, G_δ -) set U in $X \times Y$ is said to be an open (respectively closed, G_δ -) universal set of X parametrised by Y if for each open (respectively closed, G_δ -) set A of X , there is a $y \in Y$ such that $A = U^y$, where $U^y = \{x \in X: (x, y) \in U\}$.

Lemma 2. Let (X, τ) be a topological space. Then it has an open universal set parametrised by $D(|\tau|)$, the discrete space of size $|\tau|$, and one parametrised by the compact space, $2^{w(X)}$.

Proof. Let $U = \{(x, A) \in X \times \tau: x \in A\}$. Then it is clear that U is an open universal set parametrised by τ , where τ is endowed with the discrete topology.

Let $w(X) = \kappa$, and $\{B_\alpha\}_{\alpha \in \kappa}$ be a base for X . Let

$$U = \left\{ (x, f) \in X \times 2^\kappa: x \in \bigcup \{B_\alpha: f(\alpha) = 0\} \right\}.$$

Clearly all the open sets of X are the cross-sections of U . If $(x, f) \in U$, and let $\alpha \in \kappa$ such that $f(\alpha) = 0$, with $x \in B_\alpha$. Let $V = \{f \in 2^\kappa : f(\alpha) = 0\}$. Then $B_\alpha \times V$ is an open set in $X \times 2^\kappa$, containing (x, f) and is a subset of U . \square

Corollary 3. Suppose U is an open set in $X \times Y$ such that each open set of X is a countable union of the U^y 's. Then there is an open universal set for X parametrised by Y^ω .

Such a U is called a σ -generator for the open sets of X parametrised by Y . If the unions do not have to be countable, then we say that U is a generator for the open sets of X .

Lemma 4. Suppose X_i has an open universal set U_i parametrised by Y_i , for each $i \in n$. Then the set

$$U = \{(x_i)_{i \in n}, (y_i)_{i \in n} : x_i \in U_i^{y_i}, y_i \in Y_i\}$$

is a generator for the open sets of $\prod_{i \in n} X_i$ parametrised by $\prod_{i \in n} Y_i$.

Lemma 5. If U is an open (closed) universal set for X parametrised by Y , then $(X \times Y) \setminus U$ is a closed (open) universal set for X parametrised by Y .

Lemma 6. If X' is a subspace of X , then $U \cap (X' \times Y)$ is an open universal set for X' parametrised by Y .

Lemma 7. Suppose X has an open universal set U parametrised by Y , and Y' is a space. If (1) Y is a subspace of Y' or (2) Y is a continuous image of Y' , then X has an open universal set parametrised by Y' .

Proof. Ad (1): Let U' be an open set in $X \times Y'$ such that $U' \cap (X \times Y) = U$. Then U' is an open universal set parametrised by Y' .

Ad (2): Let

$$U' = \{(x, \eta) : (x, f(\eta)) \in U\} = (i_X \times f)^{-1}(U).$$

Since $i_X \times f$ is continuous, U' is an open set in $X \times Y'$. If A is an open set of X , then there is an element $y \in Y$ such that $A = U^y$. Pick $\eta \in Y'$ such that $y = f(\eta) \in Y$. Then $A = U^y = U^{f(\eta)} = (U')^\eta$. \square

Lemma 8. (1) Let X have an open universal set U parametrised by Y , and $f : X \twoheadrightarrow X'$ a continuous open surjection. Then X' also has an open universal set parametrised by Y .

(2) Let X have a closed universal set U parametrised by Y , and $f : X \twoheadrightarrow X'$ be a perfect map. Then X' has a closed universal set parametrised by Y .

Proof. In each case, let $U' = (i_X \times f)(U)$. For (2), we need to use the characterisation of perfect maps that a continuous surjection f is perfect if and only if $i_Y \times f$ is closed for all topological spaces Y . \square

2. Arbitrary spaces

Theorem 9. *The following are equivalent for a regular cardinal κ :*

- (1) $2^{\aleph_0} < 2^\kappa$;
- (2) *If X has an open universal set parametrised by Y , with $|Y| \leq \mathfrak{c}$, then $hc(X) < \kappa$;*
- (3) *Every space X with an open universal set parametrised by a compact first countable separable space Y has hereditary cellularity less than κ .*

Proof. $\neg(2) \Rightarrow \neg(1)$: Suppose X has a discrete subspace, $D(\kappa)$ of size κ . Then $D(\kappa)$ has an open universal set parametrised by Y , by Lemma 6. It is then impossible that $2^{\aleph_0} < 2^\kappa$, since there are 2^κ many open sets in $D(\kappa)$, while there are only 2^{\aleph_0} many points of Y .

$(2) \Rightarrow (3)$: Arhangel'skii's Theorem tells us that $|Y| \leq \mathfrak{c}$, for any compact first countable space Y .

$\neg(1) \Rightarrow \neg(3)$: Assume $2^{\aleph_0} = 2^\kappa$. Consider the closed unit interval $I = [0, 1]$. For each irrational number x of I , we can find a sequence of rational numbers in I , $\{x_n\}_{n \in \omega}$, converging to it. Let τ be the topology on I obtained by declaring rational numbers in I to be isolated, and the N th basic neighbourhood of an irrational point, x , of I to be $\{x\} \cup \{x_n\}_{n \geq N}$. Let Y be the space with underlying set $I \times \{0, 1\}$, and topology with base consisting of sets of the form $T \times \{0\}$ for T in τ , and $U \times \{0, 1\} \setminus (K \times \{0\})$ for U open in the usual topology on I and K a compact subset of (I, τ) . Then Y is indeed compact, separable (a countable dense set is $\mathbb{Q} \times \{0\}$) and first countable.

Now, Y has a discrete subspace, $\mathbb{P} \times \{0\}$, of size $2^{\aleph_0} = 2^\kappa$. By Lemma 2 and Lemma 7(1), since there are 2^κ many open sets in $D(\kappa)$, we deduce that $D(\kappa)$ has an open universal set parametrised by Y . \square

Theorem 10. *Suppose X has an open universal set, U , parametrised by Y . Then*

- (1) $w(X) \leq nw(Y)$;
- (2) $hd(X) \leq hL(Y)$;
- (3) $hL(X) \leq hd(Y)$;
- (4) $hc(X) \leq hc(Y)$.

Proof. *Ad (1):* Let $\mathcal{B} = \{B_\alpha\}_{\alpha \in A}$, where $|A| = w(Y)$, be a base for Y . For each $\alpha \in A$, let

$$V(B_\alpha) = \pi_X \left(\bigcup \{V \times B_\alpha : V \times B_\alpha \subseteq U \text{ and } V \text{ open}\} \right),$$

which is open in X . The collection $\{V(B_\alpha)\}_{\alpha \in A}$ is a base for X . Suppose $x \in V$ open in X . Then $V = U^y$ for some $y \in Y$, and there are $\alpha \in A$ and open $W \subseteq X$ such that $(x, y) \in W \times B_\alpha \subseteq U$, by the open-universality of U . This implies that

$$x \in \pi_X(W \times B_\alpha) = W \subseteq V(B_\alpha) \subseteq V.$$

In order to get the required inequality, we endow Y with a new topology τ' by introducing sets from the network $\mathcal{N} = \{N_\alpha\}_{\alpha \in nw(X)}$ into the original topology. Then $w(Y, \tau') = nw(Y, \tau)$ and U would still be open when Y has this topology.

The proofs of (2), (3) and (4) are very similar, so we give only give the argument for (2).

Ad (2): Suppose $hd(X) \geq \kappa$. We show that $hL(Y) \geq \kappa$. Let $\{x_\alpha\}_{\alpha \in \lambda}$ be a subset of X , and let open V_α contain x_α be such that $x_\beta \in V_\alpha$ implies $\beta \geq \alpha$. (Note that we may assume

that the V_α 's come from a basis for X .) By open-universality, there are $y_\alpha \in Y$ such that $U^{y_\alpha} = V_\alpha$. As U is open, pick open T_α and W_α such that $(x_\alpha, y_\alpha) \in T_\alpha \times W_\alpha \subseteq U$.

Take any $y_\beta \in W_\alpha$. Then $(x_\alpha, y_\beta) \in T_\alpha \times W_\alpha$, so $x_\alpha \in Y^{y_\beta} = V_\beta$. Hence β must be greater than or equal to α , and $hL(Y) \geq \lambda$. Taking the supremum of all such λ 's gives $hL(Y) \geq \kappa$. \square

Statements (2) and (3) hold because U is an open universal set. We see in [4] that if U is a G_δ -universal set, then both of them are consistently false. The proof of the theorem works equally well if U is a generator instead of an open universal set. Together with Lemma 4, we can deduce the following corollary.

Corollary 11. *Suppose X has an open universal set parametrised by Y . Then for each $n \in \omega$,*

- (1) $hd(X^n) \leq hL(Y^n)$;
- (2) $hL(X^n) \leq hd(Y^n)$;
- (3) $hc(X^n) \leq hc(Y^n)$.

Corollary 12. *For a space X , the following are equivalent:*

- (1) X has an open universal set parametrised by 2^ω ;
- (2) X has an open universal set parametrised by some separable metric space;
- (3) X has an open universal set parametrised by some cosmic space;
- (4) X is separable and metrisable.

Proof. (1) \Rightarrow (2): The space 2^ω is separable and metrisable.

(2) \Leftrightarrow (3): Note that a cosmic space is a continuous injective image of a separable metric space. Then we can use Lemma 7.

(2) \Rightarrow (4): A separable metric space has countable weight. So by Theorem 10(1), X must have countable weight.

(4) \Rightarrow (1): If X is metrisable and separable, then X has countable weight. Use Lemma 2(2). \square

A similar proof to Theorem 10(1) can be used to prove the following two propositions.

Proposition 13. *Suppose X has an open universal set U parametrised by Y . Then $\chi(X) \leq hL(Y)$. Moreover, for every compact subset K of X , $\chi(K, X) \leq hL(Y)$.*

Proof. Let $x \in X$, and $Y_x = \{y \in Y: x \in U^y\}$. For all $y \in Y_x$, there exist S_y open in X and T_y open in Y such that $(x, y) \in S_y \times T_y \subseteq U$. Then, $Y_x = \bigcup_{y \in Y_x} T_y$, which has Lindelöf degree less than or equal to $\kappa = hL(Y)$. So we can find $\{y_\alpha\}_{\alpha \in \kappa} \subseteq Y_x$ such that $Y_x = \bigcup_{\alpha \in \kappa} T_{y_\alpha}$. Then $\{S_{y_\alpha}\}_{\alpha \in \kappa}$ is a local base for x .

Let K be a compact subset of X , X' be the quotient of X having identified K to a point. Let $f: X \rightarrow X'$ be the quotient map. This map is also a perfect map. Using Lemma 8(2), we see that X' has an open universal set parametrised by Y , telling us that $\chi(X') = hL(Y)$. In particular the point $f(K)$ has character no greater than $hL(Y)$ in X' . So K must have character no greater than $hL(Y)$ in X . \square

Proposition 14. *Suppose X has an open universal set U parametrised by Y . If $hL(X \times Y) \leq \kappa$, then $w(X) \leq \kappa$.*

Proof. Let $\{A_\lambda \times B_\lambda\}_{\lambda \in \kappa}$ be basic open sets in $X \times Y$, with union equal to U . Suppose x is an element of an open set $V = U^Y$ in X . Then there is a $\lambda \in \kappa$ such that: $(x, y) \in A_\lambda \times B_\lambda \subseteq U$.

It is clear that $x \in A_\lambda \subseteq U^Y$. Hence $\{A_\lambda\}_{\lambda \in \kappa}$ is a base for X . \square

Next we use Todorćević's spaces $A[\leq]$, $A[\geq]$ and $A[\leq_{lex}]$ to show that, in the countable case, the inequalities $hd(X) \leq hd(Y^\omega)$, $hL(X) \leq hL(Y^\omega)$ and $nw(X) \leq \max(hd(Y^\omega), hL(Y^\omega))$, consistently need not hold, for a space X with an open universal set parametrised by Y .

Example 15 ($\mathfrak{b} = \aleph_1$). Let $X = A[\geq_{lex}]$ and $Y' = A[\leq_{lex}] \times \omega$. Then X has an open universal set parametrised by $Y = (Y')^\omega$, with $nw(X) > \aleph_0$ and Y being hereditarily separable and hereditarily Lindelöf.

Proof. We note that Y' is both hereditarily separable and hereditarily Lindelöf and so is any finite power of Y' . Hence Y is also hereditarily separable and hereditarily Lindelöf. As X is uncountable, it must have uncountable networkweight.

Let $U_n = \bigcup_{\alpha \in \omega_1} (B_{n, f_\alpha}[\geq_{lex} f_\alpha] \times B_{n, f_\alpha}[\leq_{lex} f_\alpha])$. Then, for each n , U_n is open and $U_n^{f_\alpha} = B_{n, f_\alpha}[\geq_{lex} f_\alpha]$. Let $U = \bigcup_{n \in \omega} (U_n \times \{n\})$. Since X is hereditarily Lindelöf, each open set in X is a countable union of the basic neighbourhoods. Therefore, U is a σ -generator for open sets parametrised by Y' , and by Corollary 3, X has an open universal set parametrised by $(Y')^\omega$. \square

Note that the above example requires an extra axiom. We now show that this must always be the case for a certain class of spaces. A space X is cometrisable if it has a coarser metric topology ν such that all points of X have a neighbourhood base of ν -closed sets. Assuming the OCA, if X is cometrisable and its square satisfies the c.c.c. hereditarily, then X is cosmic (see [6]). Note that all of the non-compact examples constructed in this paper are cometrisable.

It is known that the Proper Forcing Axiom (which implies the existence of large cardinals) implies both the OCA and (S) [5,13,11].

Theorem 16 (PFA). *Let X be cometrisable and have an open universal set U parametrised by Y . If Y satisfies the c.c.c. hereditarily, then X is metrisable.*

Proof. By Theorem 22, X^2 satisfies the hereditary c.c.c., and by the above, X is cosmic. Using (S), we see that Y must be hereditarily Lindelöf. Then $X \times Y$ is hereditarily Lindelöf, and using Proposition 14, we get X is metrisable (and separable). \square

Example 17 ($\mathfrak{b} = \aleph_1$).

(1) Let $X = A[\geq]$ and $Y' = A[\leq] \times \omega$. Then X has an open universal set parametrised by $Y = (Y')^\omega$, with X being an L -space and Y a strong S -space.

(2) Let $X = A[\leq]$ and $Y' = A[\geq] \times \omega$. Then X has an open universal set parametrised by $Y = 2^\omega \times (Y')^\omega$, with X being an S -space and Y a strong L -space.

Proof. Part (1) follows almost identically to the preceding example.

For (2), we recall that $A[\leq]$ is a Kunen line type space. Hence every open subset of X is the union of a Euclidean open set and a countable union of $A[\leq]$ basic open sets. As in Example 15, there is an open subset of $X \times (Y')^\omega$ which parametrises all countable unions of basic open subsets of $A[\leq]$. Euclidean open subsets of A can be parametrised by the Cantor set. It is now clear that Y parametrises an open universal set of $A[\leq]$. \square

Observe the above examples cannot exist if there are no S -spaces. For suppose X has an open universal set parametrised by Y . If Y is hereditarily c.c.c., then it is hereditarily Lindelöf, and X is hereditarily separable by Theorem 10, and hence hereditarily Lindelöf.

Theorem 18 (S). *Let X have an open universal set parametrised by Y . If Y is hereditarily c.c.c., then X must be both hereditarily separable and hereditarily Lindelöf.*

If we do not require the hereditary density and hereditary Lindelöf degree of the spaces X and Y to be small, there is a ZFC example, at least, one way round. Instead of A used above, we can use D^ω , where D is the discrete space of size \beth_ω (see [13, Theorem 0.5]).

Example 19. Let $X = D^\omega[\geq]$ and $Y' = D^\omega[\leq] \times \omega$. Then X has an open universal set parametrised by $Y = (Y')^\omega$, with

$$hd(X) > hL(X) = |D| = hd(Y') < hL(Y').$$

Proof. The proof follows that of Example 15. \square

Problem 20. Is there a space X with an open universal set parametrised by Y such that $hL(X) > hL(Y)$?

3. Diagonal properties

By observing a symmetry in products (Proposition 21), we can prove stronger results for spaces with G_δ -diagonals.

Proposition 21. *Suppose κ is an uncountable cardinal. Let $\{x_\alpha\}_{\alpha \in \kappa}$ be a left-separated (respectively right-separated, discrete) subset of the product space X^κ . Then there is a subset Λ of κ , with $|\Lambda| = \kappa$, and open sets V_α containing x_α ($\alpha \in \Lambda$), such that for $\alpha, \beta \in \Lambda$, if $x_\beta \in \sigma V_\alpha$ for some $\sigma \in S(n)$ then $\beta \geq \alpha$ (respectively $\beta \leq \alpha$, $\beta = \alpha$).*

Proof. Let $\kappa = \Lambda'_0$. Suppose the theorem is false. Then for any $\Lambda \in [\kappa]^\kappa$, there are $\alpha \in \Lambda$ and $\sigma \in S(n)$ such that

$$x_\alpha \in \overline{\{\sigma x_\beta : \beta < \alpha, \beta \in \Lambda\}}.$$

As $\kappa = \kappa \times \kappa$, we can consider κ to be the disjoint union of the subsets Γ_γ , $\gamma \in \kappa$, where $|\Gamma_\gamma| = \kappa$. For each γ , we can find an $\alpha_\gamma \in \Gamma_\gamma$ and a $\tau_\gamma \in S(n)$ such that $\mathbf{x}_{\alpha_\gamma} \in \overline{\{\tau_\gamma \mathbf{x}_\beta : \beta < \alpha_\gamma, \beta \in \Gamma_\gamma\}}$. By the pigeonhole principle, we can assume that for all $\gamma \in \kappa$, $\tau_\gamma = \sigma_1 \in S(n)$. Let $\Lambda_1 = \bigcup_{\gamma \in \omega_1} \Gamma_\gamma \setminus \{\alpha_\gamma\}$ and $\Lambda'_1 = \{\alpha_\gamma : \gamma \in \omega_1\}$. Note that these are two disjoint subsets of κ of size κ . For all $\beta \in \Lambda'_1$,

$$\mathbf{x}_\beta \in \overline{\{\sigma_1 \mathbf{x}_\gamma : \gamma < \beta, \gamma \in \Lambda_1\}}.$$

Now suppose for some $m > 0$ we have constructed disjoint subsets Λ_m and Λ'_m of κ of size κ , and $\sigma_m \in S(n)$ satisfying:

- $\Lambda_m \cap \Lambda'_m = \emptyset$, and $\Lambda_m \cup \Lambda'_m = \Lambda'_{m-1}$;
- for all $\beta \in \Lambda'_m$, $\mathbf{x}_\beta \in \overline{\{\sigma_m \mathbf{x}_\gamma : \gamma < \beta, \gamma \in \Lambda_m\}}$.

But just as we built the subsets Λ_1 and Λ'_1 , and picked the element $\sigma_m \in S(n)$, we can repeat the same procedure to get our subsets Λ_{m+1} and Λ'_{m+1} of Λ'_m (both of size κ), and also $\sigma_{m+1} \in S(n)$ satisfying both conditions.

Let $\pi_m = \sigma_m \circ \cdots \circ \sigma_1$, for $m \geq 1$. Now, since the group $S(n)$ is finite, there must be positive integers m and k such that $\pi_m = \pi_{m+k}$. The sequence $(\sigma_m, \sigma_{m+1}, \dots, \sigma_{m+k})$ satisfies

$$\sigma_{m+k} \circ \sigma_{m+k-1} \circ \cdots \circ \sigma_m = e.$$

To obtain the contradiction, we take an element $\alpha \in \Lambda'_{m+k}$. Then

$$\mathbf{x}_\alpha \in \overline{\{\sigma_{m+k} \mathbf{x}_\beta : \beta < \alpha, \beta \in \Lambda_{m+k}\}}.$$

Noting that for each $\beta \in \Lambda_{m+k} \subseteq \Lambda'_{m+k-1}$, $\mathbf{x}_\beta \in \overline{\{\sigma_{m+k-1} \mathbf{x}_\gamma : \gamma < \beta, \gamma \in \Lambda_{m+k-1}\}}$, we obtain

$$\mathbf{x}_\alpha \in \overline{\{\sigma_{m+k} \circ \sigma_{m+k-1} \mathbf{x}_\gamma : \gamma < \alpha, \gamma \in \Lambda_{m+k-1}\}}.$$

Repeating the above arguments, we have

$$\begin{aligned} \mathbf{x}_\alpha &\in \overline{\{\sigma_{m+k} \circ \sigma_{m+k-1} \circ \cdots \circ \sigma_m \mathbf{x}_\gamma : \gamma < \alpha, \gamma \in \Lambda_m\}} \\ &= \overline{\{\mathbf{x}_\gamma : \gamma < \alpha, \gamma \in \Lambda_m\}}. \end{aligned}$$

We have contradicted the left-separation of $\{\mathbf{x}_\alpha\}_{\alpha \in \kappa}$.

With appropriate modifications of the above arguments, the other two results of the proposition can be shown to be true. \square

Theorem 22. Suppose X is a space with a G_δ -diagonal. If X has an open universal set U parametrised by Y , then

- (1) $hL(X^{hd(Y)}) \leq hd(Y)$;
- (2) $hd(X^{hL(Y)}) \leq hL(Y)$;
- (3) $hc(X^{hc(Y)}) \leq hc(Y)$.

Proof. We only need to show the above inequalities for finite powers of X .

Ad (1): Suppose $hd(Y) = \kappa$. Note that $hL(X) \leq \kappa$, by Theorem 10. Suppose $\{x_\alpha\}_{\alpha \in \kappa^+}$ is a right-separated subset of X^n , with basic open sets $V_\alpha = \prod_{i=1}^n V_\alpha^i$, satisfying the conclusion of Proposition 21. We can assume that all of the x_α 's are off the diagonal, since $hL(X) \leq \kappa$.

Let $N = \{(m_1, m_2): m_1, m_2 = 1, \dots, n\}$, and $X^n = \prod_{m=1}^n X_m$. For $\mathbf{m} = (m_1, m_2) \in N$, let $\pi_{\mathbf{m}}$ be the projection map, and $\Delta_{\mathbf{m}}$ be the diagonal of $X_{m_1} \times X_{m_2}$. Now, X has a G_δ -diagonal, and therefore, for each \mathbf{m} , there is a countable family of open sets $\{G_k^{\mathbf{m}}\}_{k \in \omega}$ such that

$$\Delta_{\mathbf{m}} = \bigcap_{k \in \omega} G_k^{\mathbf{m}}.$$

Noting that the diagonal Δ of $\prod_{m=1}^n X_m$ is the intersection of all $\pi_{\mathbf{m}}^{-1} \Delta_{\mathbf{m}}$, we obtain

$$\Delta = \bigcap_{k \in \omega} \bigcap_{\mathbf{m} \in N} \pi_{\mathbf{m}}^{-1} G_k^{\mathbf{m}}.$$

Let $G'_k = \bigcap_{\mathbf{m} \in N} \pi_{\mathbf{m}}^{-1} G_k^{\mathbf{m}}$, for $k \in \omega$. These are open sets containing the diagonal Δ in X^n .

We can assume that there is a $k \in \omega$ such that for all $\alpha \in \kappa^+$, $x_\alpha \notin G'_k$, and $(V_\alpha^i)^n = \prod_{m=1}^n V_{\alpha, m}^i \subseteq G'_k$. Then for any $\beta \in \kappa^+$, $i = 1, \dots, n$, and $\mathbf{m} = (m_1, m_2) \in N$, $V_{\beta, m_1}^i \times V_{\beta, m_2}^i \subseteq G_k^{\mathbf{m}}$. This ensures that for all $\alpha \in \kappa^+$, it is not possible for any different x_α^i and x_α^j to lie in a V_β^k simultaneously, where $x_\alpha = (x_\alpha^1, \dots, x_\alpha^n)$.

For each $\alpha \in \kappa^+$, there is a $y_\alpha \in Y$ such that $U^{y_\alpha} = \bigcup_{i=1}^n V_\alpha^i$. Since the (x_α^i, y_α) 's are elements of the open set U , there exist open sets S_α^i in X and W_α in Y such that for each i ,

$$(x_\alpha^i, y_\alpha) \in S_\alpha^i \times W_\alpha \subseteq U.$$

Suppose that $y_\beta \in W_\alpha$. Then the (x_α^i, y_β) 's are in U , which implies that all the x_α^i 's are elements of U^{y_β} . This implies that we must have $x_\alpha \in \sigma V_\beta$, for some $\sigma \in S(n)$. Then $\alpha \leq \beta$, and that $\{y_\alpha\}_{\alpha \in \kappa^+}$ is a left-separated subset of Y , contradicting our original assumption.

Statements (2) and (3) can be proved similarly. \square

Let κ be a regular cardinal. Recall that a space X does not have a κ -accessible diagonal if for every $Y \in [X^2 \setminus \Delta]^\kappa$ there is a $Y' \in [Y]^\kappa$ such that $\overline{Y'} \cap \Delta = \emptyset$ [7]. A space X has a small diagonal if it does not have a \aleph_1 -accessible diagonal. If X has a G_δ -diagonal, then it does not have a κ -accessible diagonal for any regular cardinal κ . The above proof proves the following.

Corollary 23. *Let κ be a regular cardinal, and X be a space not having a κ -accessible diagonal. If X has an open universal set U parametrised by Y , then*

- (1) $hL(Y) \leq \kappa$ implies $hd(X^\kappa) \leq \kappa$;
- (2) $hd(Y) \leq \kappa$ implies $hL(X^\kappa) \leq \kappa$; and
- (3) $hc(Y) \leq \kappa$ implies $hc(X^\kappa) \leq \kappa$.

Example 27 is a consistent example of a (compact zero-dimensional) non-submetrisable space X with closed universal set parametrised by Y such that $hL(X) = hd(Y) = \aleph_0$ but $hL(X^2) > \aleph_0$.

4. Compact spaces

When X is compact and has an open universal set parametrised by Y , the direct implications between hereditary density and hereditary Lindelöf degree of Y and X do hold.

Lemma 24. *Let X be a compact set that has a closed universal set, U , parametrised by Y . Then:*

- (1) *the set-valued map $\Phi : y \mapsto U^y$ defined on Y is upper semi-continuous; and*
- (2) *if $Y^{(1)} = \{y \in Y : |\Phi(y)| = 1\}$ and $\phi : Y^{(1)} \rightarrow X$ be the map that picks out the unique element of $\Phi(y)$, then ϕ is a continuous map.*

Proof. Let V be an open set in X . By universality, $X \setminus V = U^y$ for some $y \in Y$. We need to show that the set

$$A = \{\eta \in Y : \Phi(\eta) \subseteq V\}$$

is open in Y . Now, A is also equal to the set $\{\eta \in Y : \Phi(\eta) \cap \Phi(y) = \emptyset\}$. Take $\eta \in A$. For all $x \in \Phi(y)$, the point (x, η) is outside the closed set U . Therefore, there exist open set S_x in X and open set T_x in Y such that

$$(x, \eta) \in S_x \times T_x \subseteq (X \times Y) \setminus U.$$

Since X is compact, $\Phi(y)$ must also be compact. Then there are x_0, \dots, x_n in X such that $\Phi(y) \subseteq \bigcup_{i=0}^n S_{x_i}$. Let $T = \bigcap_{i=0}^n T_{x_i}$. Note that $\pi_Y^{-1}(T)$ does not meet $\pi_X^{-1}(\Phi(y)) \cap U$, which means that T is an open neighbourhood containing η that is contained in A .

Let $Y^{(1)} = \{y \in Y : |\Phi(y)| = 1\}$ and $\phi : Y^{(1)} \rightarrow X$ be the map that picks out the unique element of $\Phi(y)$. The above argument shows that this is a continuous map.

Corollary 25. *Let X be a compact space that has an open universal set, U , parametrised by Y . Let \mathcal{P} be any hereditary property preserved by taking continuous images. Then, if Y^n has \mathcal{P} , so does X^n .*

Hence, in the situation above $hd(X^n) \leqslant hd(Y^n)$ and $hL(X^n) \leqslant hL(Y^n)$.

One might wonder if the existence of an open universal set for a compact space X parametrised by a hereditarily Lindelöf or hereditarily separable Y , would imply that X is in fact metrisable. Indeed if Y^2 is hereditarily Lindelöf or hereditarily separable, then either by Corollary 11 or Corollary 25, X^2 is necessarily hereditarily Lindelöf, and so X is metrisable.

Corollary 26. *A compact space with open universal set parametrised by a space whose square is either hereditarily separable or hereditarily Lindelöf, is metrisable.*

As we now see, our question has a consistent counter-example, and, provided we consider only zero-dimensional compacta, a consistent positive answer.

Example 27. It is consistent that there is a compact zero-dimensional non-metrisable space X with a closed universal set parametrised by a compact zero-dimensional S -space. Moreover, X^2 is a strong S -space.

Proof. Take the space X constructed in [2, Theorem 3.1]. Then X is compact, zero-dimensional and non-metrisable, X^2 is a strong S -space, while $\mathcal{H}(X)$, the family of all closed subsets of X with the Vietoris topology, is hereditarily separable, zero-dimensional and not hereditarily Lindelöf.

Let $Y = \mathcal{H}(X)$, and define U to be $\bigcup_{C \in \mathcal{H}(X)} C \times \{C\}$. It is straightforward to check that U is a closed subset of $X \times Y$, and hence is a closed universal set for X . \square

The authors do not have an example of a (zero-dimensional) compact non-metrisable space with an open universal set parametrised by a hereditarily Lindelöf space. Note, though, that if a space X has $\mathcal{H}(X)$ hereditarily Lindelöf, then X is compact and metrisable.

Theorem 28 (B). *Let X be compact and zero-dimensional, with a closed universal set U , parametrised by a space Y satisfying the c.c.c. hereditarily. Then X is second countable.*

Proof. Let \mathcal{C} be the Boolean algebra of all clopen subsets of X . Since we are assuming (B), it suffices to show that if \mathcal{A} is an uncountable subset of \mathcal{C} , then there are distinct A and A' in \mathcal{A} so that $A \subseteq A'$. (For then \mathcal{C} is a countable base for X .)

For each A in \mathcal{A} , pick $y_A \in Y$ such that $U^{y_A} = A$, and set $V_A = \phi^{-1}(A) = \{y \in Y^{(1)} : U^y \subseteq A\}$. According to Lemma 24(2), V_A is an open neighbourhood of y_A . As Y has the c.c.c. hereditarily, there are distinct A and A' in \mathcal{A} such that $y_A \in V_{A'}$, which occurs if and only if $A = U^{y_A} \subseteq A'$, as required. \square

Example 27 and Theorem 28 answer the question whether metrisability can be inferred from the parametrising space satisfying the c.c.c. hereditarily, in the case of the space being compact and zero-dimensional with open universal sets.

Problem 29. Can we remove the restriction to zero-dimensional compacta in the preceding result?

Problem 30. Are there ZFC counter-examples if we look at compact spaces with G_δ -universal sets?

References

- [1] J.E. Baumgartner, P. Komjáth, Boolean algebras in which every chain and antichain is countable, *Fund. Math.* 111 (1981) 125–133.
- [2] M. Bell, J. Ginsburg, S. Todorćević, Countable spread of $\exp Y$ and λY , *Topology Appl.* 14 (1982) 1–12.
- [3] R.E. Engelking, *General Topology*, Heldermann, Berlin, 1989.

- [4] P.M. Gartside, J.T.H. Lo, The hierarchy of Borel universal sets, Preprint.
- [5] G. Gruenhage, Cosmicity of metrizable spaces, *Trans. Amer. Math. Soc.* 313 (1) (1989) 301–315.
- [6] G. Gruenhage, Perfectly normal compacta, cosmic spaces and some partition problems, in: *Open Problems in Topology*, North-Holland, Amsterdam, 1990, Chapter 6.
- [7] M. Hušek, Topological spaces without κ -accessible diagonal, *Comment. Math. Univ. Carolin.* 18 (1977) 777–788.
- [8] A.S. Kechris, D.A. Martin, A note on universal sets for classes of countable G_δ 's, *Mathematika* 22 (1975) 43–45.
- [9] D.G. Larman, C.A. Rogers, The descriptive character of certain universal sets, *Proc. London Math. Soc.* (3) 27 (1973) 385–401.
- [10] A.J. Ostaszewski, Families of compact sets and their universals, *Mathematika* 21 (1974) 116–127.
- [11] J. Roitman, Basic S and L , in: *Handbook of Set-Theoretic Topology*, North-Holland, Amsterdam, 1991, Chapter 7.
- [12] S. Todorćević, Forcing positive partition relations, *Trans. Amer. Math. Soc.* 280 (2) (1983) 703–720.
- [13] S. Todorćević, Partition problems in topology, *Amer. Math. Soc. Cont. Maths.* 84 (1989).